DYNAMIC THEORY OF CONTINUOUSLY DISTRIBUTED DISLOCATIONS. ITS RELATION TO PLASTICITY THEORY

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V.L. BERDICHEVSKII and L.I. SEDOV
(Moscow)

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The theory of continuous dislocations deals with a continuous medium having a continuous distribution of defects in microstructure, namely dislocations. Below, we construct a class of models of continuous media, which comprise many known models as well as other models, including viscous, elastic and plastic effects and the motion of dislocation defects.

In particular, a variational principle, the internal energy and a dissipation function are utilized in the construction of models of plastic bodies.

In describing the distribution of dislocations in terms of a number of defining parameters(*), it is necessary to include some additional new characteristics. In view of the fact that these additional characteristics may be chosen in various ways, different theories arise. We will begin with a brief review.

In the works of Kondo, Bilby, Kröner, Sedov, Kunin and others [1 to 8] the continuum is supplied with a manifold of affine connection M, and the dislocation characteristics are given by the metric tensor, and the curvature and torsion tensors of the manifold. The manifold M may be introduced by various reasoning processes. Bilby [3 to 5] constructs the manifold M on the basis of lattice theory, and obtains, as a result, the curvature tensor equal to zero (nine new degrees of freedom). Kondo [1 and 2] defines the manifold M as the manifold of initial states which represents a metric manifold of affine connection of the most general form. The independent parameters are given by the metric tensor $g^*_{\alpha\beta}$ and the torsion tensor $S_{\alpha\beta}{}^{\gamma}$ of the manifold M (a total of 15 new parameters), while the curvature tensor is expressed in terms of their first and second derivatives [9 and 10]. Supplementary equations are necessary for $g^*_{\alpha\beta}$ and $S_{\alpha\beta}{}^{\gamma}$. In his work, Kröner [6 and 11] proposes a method of obtaining these equations, given the curvature tensor as a function of the coordinates. The case for which the curvature tensor vanishes is called the "restricted" theory. In that case, we have absolute parallelism, i.e. the basic assumptions of Bilby's theory are fulfilled. By linearizing the equations of the "restricted" theory, we obtain the equations of the so called elementary theory [11].

If the torsion tensor vanishes, then the manifold M is Riemannian, and may be imbedded in a Euclidean space E of a greater number of dimensions. By allowing the geometric characteristics of the manifold M vary with time, we obtain displacements of M in E. The components of the displacement vector of M in E may be taken as the defining parameters. Corresponding equations for reversible phenomena are obtained in [2] (see Vol. 3) by means of a variational principle.

In [12 and 13], the defining parameters are introduced without involving any concepts

*) The behavior of a medium is considered known if a system of quantities called the defining parameters are known functions of the coordinates and of time.

from differential geometry; the dynamic equations are constructed by averaging the equations are constructed by averaging the equations of motion for discrete dislocations [12]. Although the analysis confined itself to linear theory, the results obtained were in the form of extremely complex integrodifferential equations which were difficult to relate to the theory of plasticity.

Below, we will show that in order to describe the distribution of dislocations, including known plastic models, it is sufficient to introduce nine new (compared to classical elasticity) degrees of freedom (three more degrees of freedom than for the general case in the theory of plasticity).

1. Defining parameters. We consider the motion of the medium with respect to some general curvilinear reference system of an observer with space(*) coordinates x^α, time coordinate t and coordinate basis ∂_α. We also introduce a reference system(**) moving with the medium with Lagrangian coordinates ξ^μ, time coordinate t and coordinate basis $\partial\hat{}_\mu$. The motion of the medium is defined by the relation between these two reference systems [14].

$$x^\alpha = x^\alpha(\xi^\mu, t)$$

The base vectors $\partial\hat{}_\nu$ and $\partial\hat{}^\mu$ are obtained from ∂_α and ∂^β by an affine transformation (at each point)

$$\partial\hat{}_\nu = x^\alpha{}_\nu \partial_\alpha, \quad x^\alpha{}_\nu = \frac{\partial x^\alpha}{\partial \xi^\nu}; \quad \partial\hat{}^\mu = \xi^\mu{}_\beta \partial^\beta, \quad \xi^\mu{}_\beta = \frac{\partial \xi^\mu}{\partial x^\beta}; \quad \partial\hat{}_\nu = g\hat{}_{\nu\mu}\partial\hat{}^\mu$$

$$g\hat{}_{\nu\mu} = g_{\alpha\beta} x^\alpha{}_\nu x^\beta{}_\mu$$

Here, $g\hat{}_{\nu\mu}$ and $g_{\alpha\beta}$ are the covariant components of the metric tensor in the moving reference system and the observer system, respectively.

Consider two (isomorphic) groups of coordinate transformations

$$x^\alpha \to y^\beta(x^\alpha), \qquad \xi^\mu \to \eta^\nu(\xi^\mu) \tag{1.1}$$

where y^β are the new space coordinates of the observer system while η^ν are the new Lagrangian coordinates.

Thus,

$$\partial_\alpha \to \partial_\beta\, \partial x^\beta / \partial y^\alpha, \qquad \partial\hat{}^\mu \to \partial\hat{}^\nu \partial\eta^\mu / \partial\xi^\nu$$

Let $\mathbf{A}$ be an invariant of the form

$$\mathbf{A} = A^\alpha{}_\mu \partial_\alpha \partial\hat{}^\mu = C\hat{}^\nu{}_\mu \partial\hat{}_\nu \partial\hat{}^\mu$$

It is clear that $A^\alpha{}_\mu$ behave like the components of a contravariant vector for coordinate transformations of the observer reference system (for fixed μ) and like the components of a covariant vector for transformations of the moving reference system (for fixed α). The components $C\hat{}^\nu{}_\mu$ form a second order tensor for coordinate transformations of the moving reference system. The quantities $A^\alpha{}_\mu$ and $C\hat{}^\nu{}_\mu$ represent the same tensor $\mathbf{A}$, and correspond to two different choices of base vectors. In particular, for the metric tensor $\mathbf{G}$ we have the representations

$$\mathbf{G} = g_{\alpha\beta}\partial^\alpha\partial^\beta = x^\alpha{}_\mu \partial_\alpha \partial\hat{}^\mu = g\hat{}_{\mu\nu}\partial\hat{}^\mu\partial\hat{}^\nu$$

Below, we consider the derivatives of $\mathbf{A}$ with respect to the coordinates, defined by Formulas

$$\frac{\partial \mathbf{A}}{\partial x^\alpha}\partial^\alpha = \frac{\partial \mathbf{A}}{\partial \xi^\mu}\partial\hat{}^\mu = \frac{\partial A^\beta{}_\mu}{\partial x^\alpha}\partial_\beta \partial\hat{}^\mu \partial^\alpha + A^\beta{}_\mu \frac{\partial \partial_\beta}{\partial x^\alpha}\partial\hat{}^\mu\partial^\alpha + A^\beta{}_\mu \partial_\beta \frac{\partial \partial\hat{}^\mu}{\partial \xi^\nu}\xi^\nu{}_\alpha \partial^\alpha =$$

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*) The space coordinates are associated with Greek indices which range over the values of 1, 2, 3.

**) Quantities referring to the mobing system are denoted by the symbol ^.

$$= \left(\frac{\partial A^{\beta}{}_{\mu}}{\partial x^{\alpha}} + \Gamma_{\alpha\gamma}{}^{\beta} A^{\gamma}{}_{\mu} - \Gamma^{\wedge}{}_{\mu\nu}{}^{\lambda} A^{\beta}{}_{\lambda} \xi^{\nu}{}_{\alpha}\right) \ni_{\beta} \ni^{\wedge\mu} \ni^{\alpha} =$$

$$= \nabla_{\alpha} A^{\beta}{}_{\mu} \ni_{\beta} \ni^{\wedge\mu} \ni^{\alpha} = \nabla^{\wedge}{}_{\nu} A^{\beta}{}_{\mu} \ni_{\beta} \ni^{\wedge\mu} \ni^{\wedge\nu}$$

where $\Gamma_{\alpha\beta}{}^{\gamma}$ and $\Gamma^{\wedge}{}_{\mu\nu}{}^{\lambda}$ are the Christoffel symbols for the cases $\ni_{\alpha}$ and $\ni^{\wedge\mu}$, respectively.

$$\frac{\partial \ni_{\alpha}}{\partial x^{\beta}} = \Gamma_{\alpha\beta}{}^{\gamma} \ni_{\gamma}, \qquad \frac{\partial \ni^{\wedge}{}_{\mu}}{\partial \xi^{\nu}} = \Gamma^{\wedge}{}_{\mu\nu}{}^{\lambda} \ni^{\wedge}{}_{\lambda}$$

It is easily shown that x^{α}_{μ} and ξ^{μ}_{α} are constant with respect to covariant differentiation(*)

$$\nabla_{\beta} x^{\alpha}{}_{\mu} = 0, \qquad \nabla_{\beta} \xi^{\mu}{}_{\alpha} = 0$$

The time derivatives of **A** may be defined in many ways [14]. Hereafter, we will utilize the individual derivative of the following form(**):

$$DA = \frac{d}{dt}\left(A^{\alpha}{}_{\mu} \ni_{\alpha} \ni^{\wedge\mu}\right)_{\xi^{\lambda}, \ni^{\wedge\mu} = \text{const}} = \left(\frac{dA^{\alpha}{}_{\mu}}{dt} + \Gamma_{\beta\gamma}{}^{\alpha} A^{\beta}{}_{\mu} v^{\gamma}\right) \ni_{\alpha} \ni^{\wedge\mu} \tag{1.2}$$

where $v^{\gamma} = dx^{\gamma}/dt$ are the velocity components of the material points. The derivative DA is calculated for stationary Lagrangian coordinates ξ^{λ} with "stationary" Lagrangian basis $\ni^{\wedge\mu}$ and taking into account the changes in the base vectors $\ni_{\alpha}$ for a moving material point.

It is evident that, for a coordinate change, $DA^{\alpha}{}_{\mu}$ transforms like $A^{\alpha}{}_{\mu}$.

In order to construct the proposed continuous dislocation theory, it is sufficient to confine ourselves to the following collection of invariant, defining parameters:

$$\mathbf{v} = v^{\alpha} \ni_{\alpha}, \qquad \mathbf{G} = g_{\alpha\beta} \ni^{\alpha} \ni^{\beta} = x^{\alpha}{}_{\mu} \ni_{\alpha} \ni^{\wedge\mu} = g^{\wedge}{}_{\mu\nu} \ni^{\wedge\mu} \ni^{\wedge\nu}$$

$$\mathbf{A} = A^{\alpha}{}_{\mu} \ni_{\alpha} \ni^{\wedge\mu} = C^{\wedge\nu}{}_{\mu} \ni^{\wedge}{}_{\nu} \ni^{\wedge\mu}, \quad \frac{\partial \mathbf{A}}{\partial \xi^{\mu}} \ni^{\wedge\mu}, \qquad D\mathbf{A}$$

$$S, \qquad \mathbf{L}_{(p)} = L^{\wedge}{}_{(p)}{}^{\nu_1 \ldots \nu_k}{}_{\mu_1 \ldots \mu_n} \ni^{\wedge}{}_{\nu_1 \ldots \nu_k} \ni^{\wedge\mu_n} \qquad (p = 1, \ldots N) \tag{1.3}$$

where S is the entropy, while $\mathbf{L}_{(p)}$ is the collection of tensors characterizing the physical and geometric properties of the medium in the initial state (for example, anisotropy). Among the tensors $\mathbf{L}_{(p)}$, we may include the tensor

$$\mathbf{G}^{\circ} = g^{\circ}{}_{\mu\nu} \ni^{\wedge\mu} \ni^{\wedge\nu}$$

defining ds_0, the distance between particles in the initial state

$$ds_0{}^2 = g^{\circ}{}_{\mu\nu}\, d\xi^{\mu}\, d\xi^{\nu}$$

By definition

$$dL^{\wedge\nu_1 \ldots \nu_k}_{(p)\quad \mu_1 \ldots \mu_n} / dt = 0$$

Compared to classical elasticity, the number of defining parameters includes nine new degrees of freedom, namely the components of **A**. These may be given the following physical meaning.

Consider an infinitesimal particle with Lagrangian coordinates ξ^{μ}; remove it from the

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*) In [15 and 16] the covariant derivatives of x^{α}_{μ} were introduced differently, and were nonzero.

**) Here, note [14].

$$d\ni_{\alpha}/dt = \Gamma_{\alpha\beta}{}^{\gamma} \ni_{\gamma} v^{\beta}$$

body and remove the external loads. The particle will then deform, and the base vectors $\mathfrak{Э}^{\wedge}{}_{\nu}$ will become the vectors $\mathfrak{Э}_{\mu}{}^{*}$. A deformation of this type may be described by tensor **A**:

$$\mathfrak{Э}^{*}{}_{\mu} = C^{\wedge\nu}{}_{\mu}\mathfrak{Э}^{\wedge}{}_{\nu} = A^{\alpha}{}_{\mu}\mathfrak{Э}_{\alpha} \tag{1.4}$$

After deformation, the particle will be in an unstressed state, so that the components $C^{\wedge\nu}{}_{\mu}$ of **A** describe an elastic deformation(*). Moreover, from the basic condition we find that the components of **A** depend only on the coordinates ξ^{μ} and time t. In connection with **A**, we can introduce the physical parameters:

tensor of elastic deformation

$$\varepsilon^{\wedge(e)}_{\mu\nu} = {}^{1}/{}_{2}\,(g^{\wedge}{}_{\mu\nu} - g^{*}{}_{\mu\nu}), \qquad g^{*}{}_{\mu\nu} = g_{\alpha\beta}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu} \tag{1.5}$$

tensor of plastic deformation

$$\varepsilon^{\wedge(p)}_{\mu\nu} = {}^{1}/{}_{2}\,(g^{*}{}_{\mu\nu} - g^{\circ}{}_{\mu\nu}) \tag{1.6}$$

tensor of plastic deformation rates

$$e^{\wedge(p)}_{\mu\nu} = d\varepsilon^{\wedge(p)}_{\mu\nu}\,/\,dt = {}^{1}/{}_{2}\,dg^{*}{}_{\mu\nu}\,/\,dt = {}^{1}/{}_{2}g_{\alpha\beta}\,(A^{\alpha}{}_{\mu}DA^{\beta}{}_{\nu} + A^{\beta}{}_{\nu}DA^{\alpha}{}_{\mu})$$

tensor of elastic deformation gradients

$$\nabla^{\wedge}{}_{\lambda}\varepsilon^{\wedge(e)}_{\mu\nu} = -{}^{1}/{}_{2}\nabla^{\wedge}{}_{\lambda}g^{*}{}_{\mu\nu} = -{}^{1}/{}_{2}g_{\alpha\beta}\,(A^{\alpha}{}_{\mu}\nabla^{\wedge}{}_{\lambda}A^{\beta}{}_{\nu} + A^{\beta}_{\nu}\nabla^{\wedge}{}_{\lambda}A^{\alpha}{}_{\mu})$$

Consider a closed contour L within the body. It is clear that

$$\oint d\mathbf{r} = \oint \mathfrak{Э}^{\wedge}{}_{\mu}\,d\xi^{\mu} = 0$$

For the elastic deformation described above, each infinitesimal element $d\mathbf{r} = \mathfrak{Э}^{\wedge}{}_{\mu}d\xi^{\mu}$ becomes $d\mathbf{r}^{*} = \mathfrak{Э}_{\mu}{}^{*}d\xi^{\mu}$. The integral

$$\oint d\mathbf{r}^{*} = \oint \mathfrak{Э}^{*}{}_{\mu}\,d\xi^{\mu}$$

is generally nonzero and equal to a vector $\mathbf{b}_{(L)}$, which connects the ends of the broken contour L after elastic deformation(**)

$$\mathbf{b}_{(L)} = b^{\wedge\nu}_{(L)}\mathfrak{Э}^{\wedge}{}_{\nu} = \oint_{L} \mathfrak{Э}^{*}{}_{\mu}\,d\xi^{\mu} = \iint_{S} \nabla^{\wedge}{}_{[\mu}A^{\alpha}{}_{\nu]}\mathfrak{Э}_{\alpha}\,d\xi^{\mu}\,d\xi^{\nu} \tag{1.7}$$

By (1.7), each (finite) closed contour L corresponds to a vector $\mathbf{b}_{(L)}$ called the Burgers vector.

Suppose that at some point ξ^{μ} there is an infinitesimal contour surrounding the area $d\sigma$ with normal $\mathbf{n}$. Corresponding to this, there is an infinitesimal Burgers vector $\mathbf{b}_{(n)}$, which depends on ξ, $\mathbf{n}$ and $d\sigma$. In order to obtain the Burgers vector at the point ξ^{μ}, we pass to the limit

$$\lim_{d\sigma\to 0} \frac{\mathbf{b}_{(n)}}{d\sigma} = S^{\wedge\omega\lambda}n^{\wedge}{}_{\omega}\mathfrak{Э}^{\wedge}{}_{\lambda}, \qquad S^{\wedge\omega\lambda} = {}^{1}/{}_{2}\hat{\varepsilon}^{\omega\mu\nu}B^{\lambda}{}_{\alpha}\nabla^{\wedge}{}_{\mu}A^{\alpha}{}_{\nu} \tag{1.8}$$

In (1.8), $\varepsilon^{\wedge\omega\mu\nu}$ denotes the components of the alternating tensor, with $\varepsilon^{\wedge 123} = 1/\sqrt{g^{\wedge}}$; $B^{\lambda}{}_{\alpha}$ are the components of the matrix(***) which is the inverse of $A = \|A^{\alpha}{}_{\mu}\|$. Thus, the

*) The tensor in (1.4) differs from the one introduced in [6], $A = A^{\alpha}{}_{a}\mathfrak{Э}_{a}\mathfrak{Э}^{a}$ ($\alpha = 1, 2, 3$), by the rule of component transformation, since the vectors $\mathfrak{Э}^{a}$ form a different basis which does not correspond to $\mathfrak{Э}^{*}{}_{\mu}$.

**) Square brackets around indices denote an alternating operation while parenthesis around indices indicate symmetrization.

***) The positions of the indices of A^{α}_{μ} are essentially as given. Consider the lowering of indices of A^{α}_{μ} and $B^{\mu}{}_{\alpha}$ with the aid of metric tensors $g_{\alpha\beta}$ and $g^{*}{}_{\mu\nu}$ for α and μ, respectively. Since

(Footnote continued on next page)

components of the Burgers vector $b^{\mu}_{(n)}$ at a point are defined by the normal $\mathbf{n}$ and the tensor $S^{\wedge\,\omega\lambda}$ (ξ^{μ}, t), which is called the dislocation density tensor. In association with $S^{\wedge\omega\lambda}$, we consider the third-order tensor $\boldsymbol{\alpha}$ which is antisymmetric with respect to the indices $\mu\nu$

$$\boldsymbol{\alpha} = S^{\wedge}{}_{\mu\nu}{}^{\lambda}\vartheta^{\wedge\mu}\vartheta^{\wedge\nu}\vartheta^{\wedge}{}_{\lambda}, \qquad S^{\wedge}{}_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}\nabla^{\wedge}{}_{[\mu}A^{\alpha}{}_{\nu]} \tag{1.9}$$

and which is related to $S^{\wedge\,\omega\lambda}$ by the transformations

$$S^{\wedge}{}_{\mu\nu}{}^{\lambda} = \varepsilon^{\wedge}{}_{\mu\nu\omega}S^{\wedge\omega\lambda}, \qquad S^{\wedge\omega\lambda} = {}^{1}\!/_{2}\varepsilon^{\wedge\omega\mu\nu}S^{\wedge}{}_{\mu\nu}{}^{\lambda}$$

The tensor $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$ will also be called the dislocation density tensor.

If $\mathbf{b}_{(1)}$, $\mathbf{b}_{(2)}$ and $\mathbf{b}_{(3)}$ denote the Burgers vector on the surfaces whose normals are given by ϑ_1, ϑ_2 and ϑ_3, respectively, then the surface whose normal is $\mathbf{n}$ is given by

$$\mathbf{b}_{(n)} = \mathbf{b}_{(1)}n^1 + \mathbf{b}_{(2)}n^2 + \mathbf{b}_{(3)}n^3 \tag{1.10}$$

It follows from (1.8) and (1.10) that the Burgers vector is analogous to the surface load on $d\sigma$ while $S^{\wedge\,\omega\lambda}$ is analogous to the stress tensor.

We introduce $\mathbf{\Pi} = \pi^{\wedge}{}_{\mu\nu}\vartheta^{\wedge\mu}\vartheta^{\wedge\nu}$, given by the relation

$$d\vartheta^{*}{}_{\nu}/dt = \pi^{\wedge}{}_{\mu\nu}\vartheta^{*\mu} = DA^{\alpha}{}_{\nu}\vartheta_{\alpha}$$

Whence

$$\pi^{\wedge}{}_{\mu\nu} = B_{\mu\alpha}DA^{\alpha}{}_{\nu}$$

The angular velocity for the affine transformation (1.4), defined by A^{α}_{μ} will be called the plastic whirl. The corresponding antisymmetric tensor will be denoted by $\mathbf{\Omega} = \Omega^{\wedge}_{\mu\nu}\vartheta^{\wedge\mu}\vartheta^{\wedge\nu}$. It is easily seen that

$$\Omega^{\wedge}{}_{\mu\nu} = \pi^{\wedge}{}_{[\mu\nu]} \tag{1.11}$$

Note that the components of the plastic deformation rate tensor may be expressed in terms of the components of $\mathbf{\Pi}$ by Formula

$$e^{\wedge(p)}_{\mu\nu} = \pi^{\wedge}{}_{(\mu\nu)} \tag{1.12}$$

Above, in introducing the dislocation characteristics, no use was made of geometric terminology. Correspondingly, the dynamic theory, below, requires no geometric interpretation. However, in order to establish relations between this theory and the kinematic theories which are already known and to gain additional information which may facilitate a comparison between theoretical and experimental results, we will show how one may construct from $\mathbf{A}$ a metric manifold of affine connection for the "initial" state.

Define, in the moving coordinate system, the geometric quantity

$$\Gamma^{*}{}_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}\left(\frac{\partial A^{\alpha}{}_{\nu}}{\partial\xi^{\mu}} + \Gamma_{\beta\gamma}{}^{\alpha}A^{\beta}{}_{\mu}x^{\gamma}{}_{\nu}\right) \tag{1.13}$$

It is easily verified that, in going from one moving coordinate system to another, $\Gamma^{*\lambda}{}_{\mu\nu}$ transforms like the Christoffel symbols and that $g^{*}{}_{\mu\nu}$ defined in (1.5) behaves like a constant under covariant differentiation, with regard to $\Gamma^{*\lambda}{}_{\mu\nu}$, i.e.

$$\nabla^{*}{}_{\mu}g^{*}{}_{\nu\omega} = \frac{\partial g^{*}{}_{\nu\omega}}{\partial\xi^{\mu}} - \Gamma^{*}{}_{\mu\nu}{}^{\lambda}g^{*}{}_{\lambda\omega} - \Gamma^{*}{}_{\mu\omega}{}^{\lambda}g^{*}{}_{\nu\lambda} = 0 \tag{1.14}$$

The dislocation density tensor $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$ coincides with that part of the Christoffel symbol which is antisymmetric with respect to $\mu\nu$

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$$A_{\alpha\mu} \equiv g_{\alpha\beta}A^{\beta}{}_{\mu}, \qquad B_{\mu\alpha} \equiv g^{*}{}_{\mu\nu}B^{\nu}{}_{\alpha}$$

then it follows from (1.5):

$$A_{\alpha\mu} \equiv B_{\mu\alpha}$$

$$S^{\wedge}{}_{\mu\nu}{}^{\lambda} = \Gamma^*{}_{[\mu\nu]}{}^{\lambda} \tag{1.15}$$

The manifold of the "initial" state with the remaining plastic deformations is introduced as a manifold(*) with Christoffel symbol $\Gamma^*{}_{\mu\nu}{}^{\lambda}$, metric tensor $g^*{}_{\mu\nu}$ and torsion tensor $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$. It follows from (1.14) and (1.15) that $\Gamma^*{}_{\mu\nu}{}^{\lambda}$ may be expressed in terms of the metric and torsion tensors by the formula [9 and 10]

$$\Gamma^*{}_{\mu\nu}{}^{\lambda} = \left\{ {\lambda \atop \mu\nu} \right\} + S^{\wedge\cdot\cdot\lambda}_{\ \ \mu\nu\cdot} - S^{\wedge\cdot\lambda\cdot}_{\ \ \mu\cdot\nu} - S^{\wedge\cdot\lambda\cdot}_{\ \ \nu\cdot\mu} \tag{1.16}$$

where

$$\left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2} g^{*\lambda\omega} \left(\frac{\partial g^*{}_{\nu\omega}}{\partial \xi^{\mu}} + \frac{\partial g^*{}_{\mu\omega}}{\partial \xi^{\nu}} - \frac{\partial g^*{}_{\mu\nu}}{\partial \xi^{\omega}} \right)$$

Computation (taking into account (1.5) and (1.13)) shows that the curvature tensor of manifold M vanishes

$$R_{\alpha\beta\gamma}{}^{\delta} (\Gamma^*{}_{\mu\nu}{}^{\lambda},\ \partial \Gamma^*{}_{\mu\nu}{}^{\lambda} / \partial \xi^{\omega}) = 0 \tag{1.17}$$

Substitution of (1.16) into (1.17) yields the relation between the metric tensor and the density dislocation tensor. The resultant Eqs.

$$R_{\alpha\beta\gamma}{}^{\delta} \left(g^*{}_{\nu\omega},\ \frac{\partial g^*{}_{\nu\omega}}{\partial \xi^{\mu}},\ \frac{\partial^2 g^*{}_{\nu\omega}}{\partial \xi^{\mu} \partial \xi^{\lambda}},\ S^{\wedge}{}_{\mu\nu}{}^{\lambda},\ \frac{\partial S^{\wedge}{}_{\mu\nu}{}^{\lambda}}{\partial \xi^{\omega}} \right) = 0 \tag{1.18}$$

are called by some authors the fundamental geometric law. Note that, at each (fixed) point, the components of the metric tensor and the dislocation density tensor are kinematically independent, since (1.18) contains the derivatives of $g^*{}_{\mu\nu}$ and $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$ with respect to the coordinates.

By linearizing (1.18) it is possible to separate the parts of $R_{\alpha\beta\gamma}{}^{\delta}$ which depend exclusively [8] on derivatives of $g^*{}_{\mu\nu}$ from those of $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$

$$\hat{R}_{\alpha\beta\gamma}{}^{\delta} = K_{\alpha\beta\gamma}{}^{\delta} (\partial^2 g^*_{\mu\nu} / \partial \xi^{\omega} \partial \xi^{\lambda}) + N_{\alpha\beta\gamma}{}^{\delta} (\partial S^{\wedge}{}_{\mu\nu}{}^{\lambda} / \partial \xi^{\omega}) = 0 \tag{1.19}$$

The tensor $N_{\alpha\beta\gamma}{}^{\delta}$ is called the incompatibility tensor. A study of (1.19) as fundamental relations in the static, linear theory of dislocations is given in [6, 8 and 11]. Given the incompatibility tensor $N_{\alpha\beta\gamma}{}^{\delta}$, it is proposed to find the metric tensor $g^*{}_{\mu\nu}$ by use of (1.19) Such a formulation cannot be considered satisfactory, for in a realistically posed problem the incompatibility tensor itself must be determined in the solution of the problem.

Below, we obtain the dynamic equations for A^{α}_{μ} with the fundamental geometric law (1.18) satisfied identically everywhere. The tensors $g^*{}_{\mu\nu}$ and $S^{\wedge}{}_{\mu\nu}{}^{\lambda}$ are obtained from the known A^{α}_{μ} by Formulas (1.5) and (1.9).

2. Variational principle. The various models will be constructed with the aid of a variational principle [7 and 15-17]

$$\delta \int\limits_{t_1}^{t_2} \int\limits_{V} \Lambda \, d\tau \, dt + \delta W + \delta W^* = 0 \tag{2.1}$$

Here Λ is the Lagrangian, V is an arbitrary region associated with the particles of the medium, and $d\tau$ is a volume element

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*) In Kröner's restricted theory [11] with $\mathfrak{z}_{\alpha}$ as a basis for the initial state manifold, the components of the Christoffel symbols vanish. In the theory developed below, we have from (1.4)

$$\frac{\partial \mathfrak{z}^*_{\mu}}{\partial \xi^{\nu}} = \frac{\partial}{\partial \xi^{\nu}} (A^{\alpha}{}_{\mu} \mathfrak{z}_{\alpha}) = \left(\frac{\partial A^{\alpha}{}_{\mu}}{\partial \xi^{\nu}} + \Gamma_{\beta\gamma}{}^{\alpha} A^{\beta}{}_{\mu} x^{\gamma}{}_{\nu} \right) \mathfrak{z}_{\alpha} = \Gamma^*{}_{\mu\nu}{}^{\lambda} \mathfrak{z}^*{}_{\lambda}$$

$$d\tau = \sqrt{g^{\wedge}}\, d\xi^1 d\xi^2 d\xi^3 = \sqrt{g}\, dx^1 dx^2 dx^3, \qquad g^{\wedge} = \det\|g^{\wedge}_{\mu\nu}\|, \quad g = \det\|g_{\alpha\beta}\|$$

Eq. (2.1) is taken for arbitrary variations of the defining parameters which are nonzero on the boundary of the region of integration. The functional δW is the integral over the boundary of the four-dimensional region in the space of ξ^μ, t, taken with respect to a linear combination of the variations of the defining parameters, and is to be determined, while δW^* is some given functional(*). The Lagrangian Λ is a function of the tensor components in (1.3), has the dimensions of energy density, and is a scalar with respect to the transformations (1.1), we assume for simplicity that $\nabla^{\wedge}_{\nu} A^{\alpha}_{\mu}$ enter the Lagrangian only through the components of the dislocation density tensor α, defined in (1.9).

We determine the variations of the arguments of Λ (1.3) in an arbitrarily chosen but fixed observer reference system. In (2.1), by the stated conditions, the quantities varied are the trajectories of the particles of the medium

$$\delta x^\alpha = x'^\alpha(\xi^\mu, t) - x^\alpha(\xi^\mu, t)$$

The independent parameters A and the entropy S. For the variation of the components of $\mathbf{A}$, we set(**)

$$\delta \mathbf{A} = A'^{\alpha}{}_{\mu} \partial_\alpha(x')\, \hat{\partial}^\mu(\xi) - A^{\alpha}{}_{\mu}\partial_\alpha(x)\, \hat{\partial}^\mu(\xi) = \delta A^{\alpha}{}_{\mu}\partial_\alpha \partial^{\wedge\mu}$$

Under coordinate transformation, the δA^{α}_{μ} transform like the A^{α}_{μ}. For the variation of derivatives of x^α and $\mathbf{A}$, we employ Formulas

$$\delta \mathbf{v} = v'^\alpha(x', t)\,\partial_\alpha(x') - v^\alpha(x, t)\,\partial_\alpha(x) = [d(\delta x^\alpha)/dt + \Gamma_{\beta\gamma}{}^{\alpha} v^\gamma \delta x\,]\,\partial_\alpha = (D\delta x^\alpha)\,\partial_\alpha$$

$$\delta \mathbf{G} = x'^{\alpha}{}_{\mu}\partial_\alpha(x')\,\partial^{\wedge\mu}(\xi) - x^{\alpha}{}_{\mu}\partial_\alpha(x)\,\partial^{\wedge\mu}(\xi) = x^{\beta}{}_{\mu}\nabla_\beta \delta x^\alpha \partial_\alpha \partial^{\wedge\mu}$$

$$\delta(D\mathbf{A}) = \delta(DA^{\alpha}{}_{\mu}\partial_\alpha\partial^{\wedge\mu}) = (D\delta A^{\alpha}{}_{\mu})\,\partial_\alpha\partial^{\wedge\mu} = D(\delta\mathbf{A})$$

$$\delta\boldsymbol{\alpha} = (\delta S^{\wedge}{}_{\mu\nu}{}^{\lambda})\,\partial^{\wedge\mu}\partial^{\wedge\nu}\partial^{\wedge}{}_{\lambda} \tag{2.2}$$

From (1.9), we obtain

$$\delta S^{\wedge}{}_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}\nabla^{\wedge}{}_{[\mu}\delta A^{\alpha}{}_{\nu]} - \delta B^{\lambda}{}_{\alpha}\nabla^{\wedge}{}_{[\mu}A^{\beta}{}_{\nu]} = B^{\lambda}{}_{\alpha}\nabla^{\wedge}{}_{[\mu}\delta A^{\alpha}{}_{\nu]} - S^{\wedge}{}_{\mu\nu}{}^{\omega}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\omega}$$

From the relation $B^{\lambda}{}_{\beta}A^{\beta}{}_{\mu} = \delta^{\lambda}_{\mu}$, it follows that $\delta B^{\lambda}{}_{\beta} = -B^{\lambda}{}_{\alpha}B^{\mu}{}_{\beta}\delta A^{\alpha}{}_{\mu}$.

Components of the tensors $L_{(p)}$ are regarded as specified functions of ξ^μ and therefore are not varied.

In the basic Eq. (2.1), besides varying the functions (for constant ξ^μ) we also vary time t by displacement over an infinitesimal δt. For the limited purposes of the present paper such a time variation is sufficient(***). It is evident that for arbitrary A, we have

$$\delta_1 \mathbf{A} = \mathbf{A}'(\xi^\mu, t') - \mathbf{A}(\xi^\mu, t) = \delta\mathbf{A} + D\mathbf{A}\cdot\delta t$$

The symbol δ_1 will be used hereafter to denote total variation, while the symbol δ will denote variation for constant t. In particular, we can write for $\delta_1 x^\alpha$

$$\delta_1 x^\alpha = x'^\alpha(\xi^\mu, t') - x^\alpha(\xi^\mu, t) = \delta x^\alpha + v^\alpha \delta t$$

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*) The functional δW is determined, for known Λ and δW^*, for an arbitrary region Vt. As a result, we obtain the equilibrium equations. The energy-momentum tensor (in particular the stress tensor) is determined with the aid of the functional δW, and not by means of equations alone, as is generally done. The functional δW^* contains terms which take into account changes in entropy and addition of heat. This makes it possible to obtain the proper equations for reversible and irreversible processes. An equation of the form (1.2) was used by Toupin [18] and by Mindlin [19] with a fixed region Vt occupied by a body and with a given time interval $[t_1, t_2]$ to obtain models of media with reversible processes only, without taking into account the effects of heat. In that case, the functional δW is used only for the formulation of boundary conditions.

**) The arbitrariness of the variations permits their determination by various means.

***) In addition, it is assumed that t_1 and t_2 are independent of ξ^μ.

In the following, variation of the integral of Λ will mean the variation δ_1. From (2.2) and the above equations, we have (2.3)

$$\delta_1\Lambda = \delta\Lambda + \frac{d\Lambda}{dt}\delta t, \quad \delta_1 d\tau = (\nabla_\alpha \delta_1 x^\alpha)\, d\tau, \quad \frac{d}{dt}\Lambda\sqrt{g^\wedge} = \rho\sqrt{g^\wedge}\,\frac{d}{dt}\,\frac{\Lambda}{\rho}$$

Here, by definition, the density of the medium is $\rho = f(\xi^\mu)/\sqrt{g^\wedge}$. The variation of the first term in (2.1), is given by

$$\delta_1 \iint\limits_{Vt} \Lambda\, d\tau\, dt = \iint\limits_{Vt} \Big\{ X_\alpha \delta x^\alpha + \frac{\delta\Lambda}{\delta A^\alpha_{\ \nu}}\,\delta A^\alpha_{\ \nu} + \frac{\partial\Lambda}{\partial S}\,\delta S + \Big[\rho\,\frac{d}{dt}\Big(\frac{\Lambda}{\rho} - v^\alpha \frac{\partial}{\partial v^\alpha}\frac{\Lambda}{\rho} -$$

$$- DA^\alpha_{\ \mu}\frac{\partial}{\partial(DA^\alpha_{\ \mu})}\frac{\Lambda}{\rho}\Big) + \nabla_\beta(\sigma_\alpha{}^\beta v^\alpha + \sigma^\wedge{}^{\nu\mu\lambda}\pi^\wedge{}_{\nu\mu}x^\beta{}_\lambda)\Big]\,\delta t\Big\}\, d\tau\, dt\, -$$

$$- \iint\limits_{\Sigma t} (\sigma_\alpha{}^\beta \delta_1 x^\alpha + \sigma^\wedge{}_\nu{}^{\mu\lambda} x^\beta{}_\lambda B^\nu{}_\alpha \delta_1 A^\alpha{}_\mu)\, n_\beta\, d\sigma\, dt\, -$$

$$- \int\limits_V \rho\Big[\delta_1 x^\alpha \frac{\partial}{\partial v^\alpha}\frac{\Lambda}{\rho} + \delta_1 A^\alpha{}_\nu \frac{\partial}{\partial(DA^\alpha{}_\nu)}\frac{\Lambda}{\rho}\Big]_{t_1}^{t_2} d\tau \qquad (2.4)$$

Here, n_β are the components of the vector normal to the boundary of the space region V, namely the surface Σ. In (2.4), we have introduced the following notation:

$$\sigma_\alpha{}^\beta = -x^\beta{}_\mu \frac{\partial\Lambda}{\partial x^\alpha{}_\mu} - \Lambda\delta^\beta_\alpha, \quad \sigma^\wedge{}_\nu{}^{\mu\lambda} = -\frac{\partial\Lambda}{\partial S^\wedge{}_{\lambda\mu}{}^\nu}, \quad X_\alpha = -\rho D\frac{\partial\Lambda/\rho}{\partial v^\alpha} + \nabla_\beta\sigma_\alpha{}^\beta$$

$$\frac{\delta\Lambda}{\delta A^\alpha{}_\mu} = -\rho D\frac{\partial\Lambda/\rho}{\partial(DA^\alpha{}_\mu)} - \nabla^\wedge{}_\nu\,\frac{\partial\Lambda}{\partial(\nabla^\wedge{}_\nu A^\alpha{}_\mu)} + \frac{\partial\Lambda}{\partial A^\alpha{}_\mu} = \qquad (2.5)$$

$$= -\rho D\frac{\partial\Lambda/\rho}{\partial(DA^\alpha{}_\mu)} + \nabla^\wedge{}_\nu(\sigma^\wedge{}_\lambda{}^{\mu\nu}B^\lambda{}_\alpha) + \sigma^\wedge{}_\lambda{}^{\omega\nu}S^\wedge{}_{\nu\omega}{}^\mu B^\lambda{}_\alpha + \frac{\partial\Lambda}{\partial A^\alpha{}_\mu}$$

3. Basic equations. Further, by definition, we set

$$\delta W^* = \iint\limits_{Vt}\Big\{\rho\Theta\delta S + F_\alpha\delta_1 x^\alpha - \tau_\alpha{}^\beta\nabla_\beta\delta x^\alpha -$$

$$- Q^{\wedge\mu\nu}B_{\mu\alpha}\delta A^\alpha{}_\nu - Q^{\wedge\mu\nu\lambda}\nabla^\wedge{}_\lambda(B_{\mu\alpha}\delta A^\alpha{}_\nu) + N\delta t)\Big\}\, d\tau\, dt =$$

$$= \iint\limits_{Vt}\Big\{\rho\Theta\delta S + (F_\alpha + \nabla_\beta\tau_\alpha{}^\beta)\,\delta x^\alpha + (-Q^{\wedge\mu\nu} + \hat\nabla_\lambda\hat Q^{\mu\nu\lambda})\,B_{\mu\alpha}\delta A^\alpha{}_\nu + [F_\alpha v^\alpha + N +$$

$$+ \nabla_\beta(\tau_\alpha{}^\beta v^\alpha + Q^{\wedge\mu\nu\lambda}\pi^\wedge{}_{\mu\nu}x^\alpha{}_\lambda)]\,\delta t\}\, d\tau\, dt\, -$$

$$- \iint\limits_{\Sigma t}\{\tau_\alpha{}^\beta\delta_1 x^\alpha + Q^{\wedge\nu\mu\lambda}x^\beta{}_\lambda B_{\nu\alpha}\delta_1 A^\alpha{}_\mu\}\, n_\beta\, d\sigma\, dt \qquad (3.1)$$

Here Θ is a scalar, which for most models is the absolute temperature, the quantities

$$F_\alpha,\ \tau_\alpha{}^\beta,\ Q^{\wedge\mu\nu},\ Q^{\wedge\mu\nu\lambda},\ N$$

are generalized loads and stresses defining for a small particle the external action and the internal irreversible effects. First, setting equal to zero the total variations δ_1 of the variables on the boundary of the four-dimensional region Vt (here by definition $\delta W = 0$) we obtain from the basic variational Eq. (2.1), by taking into account (2.4), (2.5) and (3.1), the following system of equations

$$-\rho DJ_\alpha + \nabla_\beta p_\alpha{}^\beta + F_\alpha = 0 \qquad (3.2)$$

$$\rho \frac{d}{dt}\left(\frac{\Lambda}{\rho} - v^{\alpha} J_{\alpha} - A^{\beta\mu} D A^{\alpha}{}_{\mu} J_{\alpha\beta}\right) + \tag{3.3}$$
$$+ \nabla_{\beta} (p_{\alpha}{}^{\beta} v^{\alpha} + q^{\wedge\mu\nu\lambda} \pi^{\wedge}{}_{\mu\nu} x^{\beta}{}_{\lambda}) + Q^{\wedge\mu\nu} \pi^{\wedge}_{\mu\nu} + N + F_{\alpha} v^{\alpha} = 0$$

$$\frac{\partial \Lambda}{\partial S} + \rho\Theta = 0, \qquad \frac{\delta\Lambda}{\delta A^{\alpha}{}_{\nu}} A^{\alpha\mu} = Q^{\wedge\mu\nu} - \nabla^{\wedge}{}_{\lambda} Q^{\wedge\mu\nu\lambda} \tag{3.4}$$

$$p_{\alpha}{}^{\beta} = \sigma_{\alpha}{}^{\beta} + \tau_{\alpha}{}^{\beta}, \qquad q^{\wedge\mu\nu\lambda} = \sigma^{\wedge\mu\nu\lambda} + Q^{\wedge\mu\nu\lambda} \tag{3.5}$$
$$J_{\alpha} = \frac{\partial}{\partial v^{\alpha}} \frac{\Lambda}{\rho}, \qquad J_{\alpha\beta} = A_{\beta\mu} \frac{\partial}{\partial (DA^{\alpha}{}_{\mu})} \frac{\Lambda}{\rho}$$

Introducing arbitrary variations different from zero on the boundary, we can obtain Expression

$$\delta W = \iint\limits_{\Sigma t} (p_{\alpha}{}^{\beta} \delta_1 x^{\alpha} + q^{\wedge\mu\nu\lambda} x^{\beta}{}_{\lambda} B_{\mu\alpha} \delta_1 A^{\alpha}{}_{\nu}) n_{\beta}\, d\sigma\, dt +$$
$$+ \int\limits_{V} \rho\, [J_{\alpha} \delta_1 x^{\alpha} + J_{\alpha\beta} A^{\beta\mu} \delta_1 A^{\alpha}{}_{\mu}]_{t_1}^{t_2} d\tau \tag{3.6}$$

Eqs. (3.2) are momentum equations; (3.3) is the energy equation. The system of Eqs. (3.2) to (3.4) together with the Eqs. of state (2.6) and (3.5) form a closed system if Λ and δW^* are given (i.e. Λ, F_{α}, $\tau_{\alpha}{}^{\beta}$, $Q^{\wedge\mu\nu}$, $Q^{\wedge\mu\nu\lambda}$ and N are given).

In the formulation of actual problems, in order to obtain a solution it is necessary to have given the functional δW on the boundary of the medium as well, which leads to the following boundary conditions for $t = t_1$ and $t = t_2$:

$$p_{\alpha}{}^{\beta} n_{\beta} = T_{\alpha}, \qquad q^{\wedge\mu\nu\lambda} x^{\gamma}{}_{\lambda} n_{\gamma} = q^{*\mu\nu} \qquad \text{на } \Sigma \tag{3.7}$$
$$J_{\alpha} = J_{\alpha}{}^{1}, \quad J_{\alpha\beta} = J_{\alpha\beta}^{1} \quad \text{for } t = t_1; \qquad J_{\alpha} = J^{2}{}_{\alpha}, \quad J_{\alpha\beta} = J_{\alpha\beta}^{2} \quad \text{for } t = t_2$$

With the aid of (2.1), we can also obtain conditions at jumps [20].

If we assume that the Lagrangian Λ is equal to the difference between the kinetic energy and internal energy

$$\Lambda = T - \rho U \tag{3.8}$$

then the equations of motion (4.2) may be reduced to the usual form

$$\rho \frac{dv_{\alpha}}{dt} = \dot{\nabla}_{\beta} p_{\alpha}{}^{\beta} + F_{\alpha} \tag{3.9}$$

From (3.6) and (3.9), it follows that $p_{\alpha}{}^{\beta}$ is a stress tensor component, and F_{α} is a component of the body force vector. Utilizing Eqs.

$$\partial\rho / \partial x^{\alpha}{}_{\mu} = -\rho \xi^{\mu}{}_{\alpha}$$

and taking into account (2.5) and (3.5) for $p_{\alpha}{}^{\beta}$, we obtain

$$p_{\alpha}{}^{\beta} = x^{\beta}{}_{\mu} \frac{\partial U}{\partial x^{\alpha}{}_{\mu}} + \tau_{\alpha}{}^{\beta} \tag{3.10}$$

We now return to the previously obtained Euler Eqs. (3.4) for the internal degrees of freedom associated with the components $A^{\alpha}{}_{\mu}$. After some obvious transformations and taking into account (3.5) and (3.4) may be reduced to the form

$$\rho D J_{\alpha\beta} = \nabla_{\gamma} K_{\alpha\beta}{}^{\gamma} + \rho h_{\alpha\beta}, \qquad K_{\alpha\beta}{}^{\gamma} = q^{\wedge\mu\nu\lambda} A_{\alpha\nu} A_{\beta\mu} x^{\gamma}{}_{\lambda} \tag{3.11}$$

$$\rho h_{\alpha\beta} = \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}} A_{\beta\mu} + \frac{\partial\Lambda}{\partial (DA^{\alpha}{}_{\mu})} DA_{\beta\mu} - Q^{\wedge\mu\nu} A_{\alpha\mu} A_{\beta\nu} - Q^{\wedge\mu\nu\lambda} \nabla^{\wedge}{}_{\lambda} (A_{\alpha\mu} A_{\beta\nu})$$

Eqs. (3.11), antisymmetrized with respect to $\alpha\beta$, are the equations of internal moment of momentum. The quantities $J_{[\alpha\beta]}$ may be regarded as the components of the internal moment of momentum tensor; $K_{[\alpha\beta]}{}^{\gamma}$ are the components of the tensor of surface couples, and $h_{[\alpha\beta]}$ are the components of the tensor of internal body couples.

4. Equation of entropy balance. Phenomenological theory of irreversible processes. Consider the energy Eq. (3.3). With the aid of (3.1) and (3.4), it may be transformed into

$$\rho\Theta\frac{dS}{dt} = N + Q^{\wedge\mu\nu}\pi^{\wedge}{}_{\mu\nu} + Q^{\wedge\mu\nu\lambda}\nabla_{\lambda}{}^{\wedge}\pi^{\wedge}{}_{\mu\nu} + \tau^{\wedge\mu\nu}\nabla^{\wedge}{}_{\nu}v^{\wedge}{}_{\mu} \qquad (4.1)$$

Eq. (4.1) is the equation of entropy balance and may serve as a means of specifying the generalized forces and stresses in δW^*. We assume that the entropy increase connected with N is specified only by the flow of heat $\mathbf{q}$ and that N satisfies Eq. $N = -\operatorname{div}\mathbf{q}$.

Internal production of entropy $\rho\Theta d_i S$ will be denoted by σdt and, from the definition, we determine σ from the right-hand side of (4.1) as follows:

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 \qquad (4.2)$$

$$\sigma_1 = -\Theta^{-1}q^{\wedge\mu}\nabla^{\wedge}{}_{\mu}\Theta, \qquad \sigma_2 = Q^{\wedge\mu\nu}\pi^{\wedge}{}_{\mu\nu} + Q^{\wedge\mu\nu\lambda}\nabla^{\wedge}{}_{\lambda}\pi^{\wedge}{}_{\mu\nu}, \quad \sigma_3 = \tau^{\wedge\mu\nu}\nabla^{\wedge}{}_{\nu}v^{\wedge}_{\mu}$$

The quantity σ_1 characterizes irreversible effects arising from heat flow, σ_2 results from plastic deformation, and σ_3 results from viscous dissipation. In accordance with the second law of thermodynamics, σ satisfies the inequality $\sigma \geqslant 0$, while σ_1, σ_2 and σ_3 may in general be positive or negative.

As a basic assumption, we consider σ to be a function of thermodynamic fluxes,

$$\nabla^{\wedge}{}_{\mu}\Theta, \ \pi^{\wedge}{}_{\mu\nu}, \ \nabla^{\wedge}{}_{\lambda}\pi^{\wedge}{}_{\mu\nu}, \ \nabla^{\wedge}{}_{\nu}v^{\wedge}{}_{\mu}$$

the system of quantities in (2.3) and some additional (constant or variable) parameters χ_s which appear(*) in the specification of δW^*.

In addition, it is understood that χ_s are given functionals of the defining parameters (1.3). In accordance with the general theory of irreversible processes [21 and 22] we specify the functional dependence of σ (or σ_1, σ_2, and σ_3) on its arguments, and consider that for irreversible processes the generalized forces $q^{\wedge\mu}$, $Q^{\wedge\mu\nu}$, $Q^{\wedge\mu\nu\lambda}$ and $\tau^{\wedge\mu\nu}$ are related to the thermodynamic fluxes by relations of the form

$$-\Theta^{-1}q^{\wedge\mu} = \mu_1\frac{\partial\sigma}{\partial(\nabla^{\wedge}{}_{\mu}\Theta)}, \quad Q^{\wedge\mu\nu} = \mu_2\frac{\partial\sigma}{\partial\pi^{\wedge}{}_{\mu\nu}}, \quad Q^{\wedge\mu\nu\lambda} = \mu_2\frac{\partial\sigma}{\partial\pi^{\wedge}{}_{\mu\nu}}$$

$$\tau^{\wedge\mu\nu} = \mu_3\frac{\partial\sigma}{\partial(\nabla^{\wedge}{}_{\nu}v^{\wedge}{}_{\mu})} \qquad (4.3)$$

Here, the partial derivatives are taken with $\chi_s = \text{const.}$ and μ_1, μ_2 and μ_3 as coefficients. Instead of postulating (4.3), we could have taken other equivalent assumptions, such as for example, those analogous to Drucker's [23] postulate and the hypotheses of Ziegler [21], Prigogine [24], etc. The addition of χ_s and utilization of various coefficients μ_1, μ_2', μ_3 for different thermodynamic fluxes in (4.3) results in a relaxation of the usually accepted hypothesis.

Besides σ_1, σ_2 and σ_3, it is necessary to specify the coefficients μ_1, μ_2, μ_3 so as to satisfy Eq.

$$\sigma = \mu_1\gamma_1 + \mu_2\gamma_2 + \mu_3\gamma_3, \qquad \gamma_1 = \nabla^{\wedge}{}_{\mu}\Theta\frac{\partial\sigma}{\partial(\nabla^{\wedge}{}_{\mu}\Theta)} \qquad (4.4)$$

$$\gamma_2 = \pi^{\wedge}{}_{\mu\nu}\frac{\partial\sigma}{\partial\pi^{\wedge}{}_{\mu\nu}} + \nabla^{\wedge}{}_{\lambda}\pi^{\wedge}{}_{\mu\nu}\frac{\partial\sigma}{\partial(\nabla^{\wedge}{}_{\lambda}\pi^{\wedge}{}_{\mu\nu})}, \qquad \gamma_3 = \nabla^{\wedge}{}_{\nu}v^{\wedge}{}_{\mu}\frac{\partial\sigma}{\partial(\nabla^{\wedge}{}_{\nu}v^{\wedge}{}_{\mu})}$$

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*) In particular examples of known models in the theory of plasticity, χ_s are given by the following

$$\chi_1 = \int p^{\alpha\beta}\,de^{(p)}_{\alpha\beta}, \qquad \chi_2 = \int\sqrt{de^{(p)}_{\alpha\ \beta}\ de^{(p)\alpha\beta}}$$

The preceding theory is developed on the assumption that the Lagrangian Λ is independent of χ_s.

In particular, if the thermodynamic fluxes $\nabla^{\wedge}_{\mu}\Theta$ appear in σ_1 only, $\pi^{\wedge}_{\mu\nu}$ and $\nabla^{\wedge}_{\lambda}\pi^{\wedge}_{\mu\nu}$ in σ_2 only and $\nabla^{\wedge}_{\nu}v^{\wedge}_{\mu}$ in σ_3 only, then μ_1, μ_2, μ_3 satisfy the relations

$$\mu_1\gamma_1 = \sigma_1, \qquad \mu_2\gamma_2 = \sigma_2, \qquad \mu_3\gamma_3 = \sigma_3 \tag{4.5}$$

From (4.2), it follows that the absence of dissipation is related to Eq. $\sigma = 0$. Physically, it is clear that for

$$\nabla^{\wedge}_{\mu}\Theta = 0, \qquad \pi^{\wedge}_{\mu\nu} = 0, \qquad \nabla^{\wedge}_{\lambda}\pi^{\wedge}_{\mu\nu} = 0, \qquad \nabla^{\wedge}_{\nu}v^{\wedge}_{\mu} = 0 \tag{4.6}$$

each of the quantities σ_1, σ_2, σ_3 also vanishes. The converse is related to the properties of the functions σ_1, σ_2, σ_3 used in the construction of the models. In the general case, Eqs. (4.6) do not necessarily follow from $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

Relations (4.3) depend essentially on the presence of irreversible dissipative processes for $\sigma_\alpha \neq 0$. For reversible processes, the second and third Eqs. in (4.3) must be replaced by the corresponding relations for $\pi^{\wedge}_{\mu\nu}$ and $\nabla^{\wedge}_{\lambda}\pi^{\wedge}_{\mu\nu}$ in reversible processes.

Below, it is shown that by means of the method developed here we can construct and generalize many of the models used in applications, in particular models of plastic media. In each case, the $\sigma_\alpha(\alpha = 1, 2, 3)$ are chosen as homogeneous functions of the corresponding thermodynamic fluxes raised to some power k_α. The coefficients μ_α are then determined from (4.5), and are constants: $\mu_\alpha = k_\alpha^{-1}$. In particular, if σ is a quadratic form in all of its arguments, then $\mu_1 = \mu_2 = \mu_3 = 1/2$, and the relation between thermodynamic forces and fluxes will be linear. In that case, the Onsager relations follow from (4.3).

Models for plastic media cannot be obtained by utilizing linear relations between thermodynamic forces and fluxes. In that case, the dissipation function σ_2 may be chosen as a homogeneous function of first degree ($\mu_2 = 1$) with respect to $\pi^{\wedge}_{\mu\nu}$, $\nabla^{\wedge}_{\lambda}\pi^{\wedge}_{\mu\nu}$. It is clear that in this case the tensor components $Q^{\wedge\mu\nu}$ and $Q^{\wedge\mu\nu\lambda}$ will lie on some surfaces within the space of the variables $\{Q^{\wedge\mu\nu}, Q^{\wedge\mu\nu\lambda}\}$

$$f_k(Q^{\wedge\mu\nu}, \ Q^{\wedge\mu\nu\lambda}, \ \chi_s) = 0, \qquad (k = 1, \ \ldots, \ m < 36) \tag{4.7}$$

which may be called the loading surfaces(*). Indeed in the case under consideration, $Q^{\wedge\mu\nu}$ $Q^{\wedge\mu\nu\lambda}$ as determined from (4.3) will be homogeneous functions of degree zero, and consequently will not depend on all the quantities $\pi^{\wedge}_{\mu\nu}$ and $\nabla^{\wedge}_{\lambda}\pi^{\wedge}_{\mu\nu}$ but only on the independent arguments of the form $\pi^{\wedge}_{22}/\pi^{\wedge}_{11}$,..., the number of which is one less than the number of tensor components $Q^{\wedge\mu\nu}$, $Q^{\wedge\mu\nu\lambda}$. Hence, there must exist at least one relationship of the form (4.7) among the generalized forces(**).

The entire preceding theory was developed within the confines of finite deformation

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*) In the case of classical models in the theory of plasticity for $k = 1$, the relation between the dissipation function and the loading function was investigated by Ivlev [25].

**) If $\sigma(x^i)$ is a homogeneous function of first degree, then there exists a relationship among the generalized stresses, which are defined by Formulas

$$X_i = \frac{\partial\sigma}{\partial x^i} = \varphi_i(\nu^j), \qquad \nu^j = \frac{x^j}{x^1} \qquad \begin{pmatrix} i = 1, 2, \ldots, n \\ j = 2, 3, \ldots, n \end{pmatrix}$$

Indeed, assuming that these relations for $i = 2,\ldots, n$ are solved for ν^j, then we can write

$$\nu^j = \psi^j(X_2,\ldots,X_n)$$

Substituting these functions into the equation with $i = 1$

$$X_1 - \varphi_1(\nu^j) = X_1 - \varphi_1(\psi^j(X_2,\ldots,X_n)) = f(X_i) = 0 \quad (A)$$

we obtain the desired relation $f(X_i) = 0$ which interrelates the stresses X_i. This relation may be regarded as an equation of a stress surface in the X_i space. If there are only $s < n - 1$ independent functions among n functions $\varphi_i(\nu^2,\ldots, \nu^n)$ then the above stated assumption (pertaining to $s = n - 1$) concerning the possibility of obtaining a solution does not hold. In that case, instead of one equation we have $n - s$ equations of the form

$$f_k(X_i) = 0 \qquad (k = 1,2,\ldots, \quad n - s)$$

theory; assumptions regarding small defomations can only lead to simplification or linearization of the previously obtained relations.

5. Classical models. 1°. Ideal fluid model (gas). Here $\Lambda = 1/2\rho v^2 - \rho U$, whereupon the internal energy is a function of density ρ and entropy S only, and the functional δW^* takes the form

$$\delta W^* = \int_V \int_t [\rho\Theta\delta S + F_\alpha \delta_1 x^\alpha + N\delta t]\, d\tau\, dt \tag{5.1}$$

The stress tensor is obtained from (3.10), and turns out to be spherical

$$p_\alpha{}^\beta = -p\delta_\alpha{}^\beta, \qquad p = \rho^2 \partial U / \partial\rho$$

The Euler Eqs. (4.2) become the equations of motion of an ideal fluid. Further, we obtain

$$\Theta = (\partial U / \partial S)_{\rho=\text{const}}$$

For a given N, (5.1) yields the defining Eq. for S

$$\rho\Theta\, dS / dt = N$$

In that case, in solving problems we may obtain and investigate the residual stresses which are reversible by nature.

2°. Model of a viscous, heat-conducting fluid (gas). The differences from the ideal fluid are only in the composition of the functional δW^*. For viscous fluid models, we set

$$\delta W^* = \int_V \int_t [\rho\Theta\delta S + F_\alpha \delta_1 x^\alpha - \tau_\alpha{}^\beta \nabla_\beta \delta x^\alpha - \operatorname{div} \mathbf{q}\delta t]\, d\tau\, dt$$

The entropy balance equation in this case becomes

$$\rho\Theta \frac{dS}{dt} = -\Theta \operatorname{div} \frac{\mathbf{q}}{\Theta} - \frac{\mathbf{q}}{\Theta} \operatorname{grad} \Theta + \tau^{\wedge\mu\nu} \nabla^\wedge{}_\nu v^\wedge{}_\mu$$

If we assume that there exists a dissipation function

$$\sigma(\nabla^\wedge{}_\mu\Theta, \nabla^\wedge{}_\mu v^\wedge{}_\nu, \mathbf{L}_{(p)}) = -\frac{\mathbf{q}}{\Theta} \operatorname{grad} \Theta + \tau^{\wedge\mu\nu} \nabla^\wedge{}_\nu v^\wedge{}_\mu$$

where $\mathbf{L}_{(p)}$ are quantities defined in Section 2, then, taking into account (4.3), we obtain the following relations:

$$-\frac{1}{\Theta} q^{\wedge\mu} = \mu_1 \frac{\partial\sigma}{\partial(\nabla^\wedge{}_\mu\Theta)}, \qquad \tau^{\wedge\mu\nu} = \mu_3 \frac{\partial\sigma}{\partial(\nabla^\wedge{}_\nu v^\wedge{}_\mu)} \tag{5.2}$$

If the dissipation function depends on $\nabla^\wedge{}_\nu v^\wedge{}_\mu$ only through the components of the deformation rate tensor, then $\tau^{\wedge\mu\nu}$ is symmetrical.

The classical Navier-Stokes model of a viscous medium is obtained by assuming that the dissipation function σ is a positive definite quadratic form in $\nabla^\wedge{}_\mu\Theta$ and $\nabla^\wedge{}_\nu \mu^\wedge{}_\mu$. In this case, the relations in (5.2) are linear and conform to Onsager's principle.

The assumption of isotropy introduces substantial simplifications in the quadratic form for σ, whereupon the coupling between temperature and viscosity effects disappears.

3°. Elastic body model. In this case, we have

$$\Lambda = {}^1\!/_2\rho v^2 - \rho U(\varepsilon^{\wedge(e)}_{\mu\nu}, S, L_{(p)}), \qquad \varepsilon^\wedge{}_{\mu\nu}^{(e)} = {}^1\!/_2(g^\wedge{}_{\mu\nu} - g^*{}_{\mu\nu})$$

$$\delta W^* = \int_V \int_t [\rho\Theta\delta S + F_\alpha \delta_1 x^\alpha + N\delta t]\, d\tau\, dt$$

To obtain all the relations for a particular case of the system of equations given in Sections 3 and 4, we set

$$g^*{}_{\mu\nu} = g^\circ{}_{\mu\nu}(\xi^\lambda), \qquad A^\alpha{}_\mu(\xi^\lambda) = x^\alpha{}_\mu(\xi^\lambda, t_0)$$

Since A^α_μ defines the transformation from the observer coordinates x^α to the fixed basis of the "initial" state", the A^α_μ are excluded from the number of unknown variables and become parameters of the type $\mathbf{L}_{(p)}$.

The momentum equations are reduced to (3.9), wherein in accordance with (3.10) the stress tensor components $p^{\gamma\beta} = g^{\gamma\alpha} p_\alpha{}^\beta$ may be written in the form

$$p^{\gamma\beta} = \rho \frac{\partial U}{\partial x^\alpha{}_\mu} x^\beta{}_\mu g^{\gamma\alpha} = \rho \frac{\partial U}{\partial \varepsilon^\wedge{}_{\mu\nu}^{(e)}} x^\gamma{}_\mu x^\beta{}_\nu = p^{\wedge\mu\nu} x^\gamma{}_\mu x^\beta{}_\nu$$

The formula for Θ and the energy equation, equivalent to the entropy balance , may be written in the form

$$\Theta = \left(\frac{\partial U}{\partial S}\right)_{\varepsilon^\wedge{}_{\mu\nu}^{(e)}=\text{const}}, \quad \rho\Theta \frac{dS}{dt} = N$$

4°. Models of plastic bodies. With some additional assumptions, many known models of the theory of ideal plasticity and plasticity with hardening may be obtained from the general theory developed in Sections 3 and 4. For this purpose, it is sufficient to take

$$\Lambda = {}^1/_2\rho v^2 - \rho U\,(g^*{}_{\mu\nu},\ g^\wedge{}_{\mu\nu} S,\ L_{(p)})$$
$$\delta W^* = \int\limits_V \int\limits_t [\rho\Theta\delta S + F_\alpha \delta_1 x^\alpha - Q^{\wedge\mu\nu}\delta\varepsilon^\wedge{}_{\mu\nu}^{(p)} + N\delta t]\, d\tau\, dt \tag{5.4}$$

In contrast with the general case, the components of $A^\alpha{}_\mu$ enter Λ and δW^* only through the components of the metric tensor in the initial state

$$g^*{}_{\mu\nu} = g_{\alpha\beta} A^\alpha{}_\mu\, A^\beta{}_\nu, \quad [\varepsilon^\wedge{}_{\mu\nu}^{(p)} = {}^1/_2\,(g^*{}_{\mu\nu} - g^\circ{}_{\mu\nu})]$$

Clearly $Q^{\wedge\mu\nu}$ may be considered to be symmetric; in applying the general theory, it is necessary to take into account the identity

$$Q^{\wedge\mu\nu}\delta\varepsilon^\wedge{}_{\mu\nu}^{(p)} = Q^{\wedge\mu\nu} B_{\nu\alpha}\delta A^\alpha{}_\mu$$

The arbitrariness of the variations $\delta\varepsilon^\wedge{}_{\mu\nu}^{(p)} = {}^1/_2\ \delta g^*{}_{\mu\nu}$ now leads to only six equations instead of the nine equations in (3.4):

$$-\rho \frac{\partial U}{\partial g^*{}_{\mu\nu}} = {}^1/_2 Q^{\wedge\mu\nu} \tag{5.5}$$

From (3.10) as well as (5.3), we obtain the stress tensor.

$$p^{\wedge\mu\nu} = \rho \frac{\partial U}{\partial \varepsilon^\wedge{}_{\mu\nu}^{(e)}} \tag{5.6}$$

In the important particular case, when the components $g^*{}_{\mu\nu}$ and $g^\wedge{}_{\mu\nu}$ in (5.4) appear only as differences $g^\wedge{}_{\mu\nu} - g^*{}_{\mu\nu} = 2\varepsilon^\wedge{}_{\mu\nu}^{(e)}$, (a medium "with no memory") we obtain

$$p^{\wedge\mu\nu} = Q^{\wedge\mu\nu} \tag{5.7}$$

It should be stressed that (5.7) does not hold for a medium "with memory", when the internal energy depends on the components of elastic deformation as well as the components of plastic deformation $\varepsilon^\wedge{}_{\mu\nu}^{(p)}$. In accordance with (4.2) σ_2 is given by

$$\sigma_2 = Q^{\wedge\mu\nu} e^\wedge{}_{\mu\nu}^{(p)} \tag{5.8}$$

If (5.7) holds, then (5.8) yields

$$\sigma_2 = p^{\wedge\mu\nu} e^\wedge{}_{\mu\nu}^{(p)} \tag{5.9}$$

However, for the general case, (5.9) does not hold. For a medium "with memory", (5.5) may be written in the form

$$-2\rho \frac{\partial U}{\partial g^*{}_{\mu\nu}} = \rho \frac{\partial U}{\partial \varepsilon^\wedge{}_{\mu\nu}^{(e)}} - \rho \frac{\partial U}{\partial \varepsilon^\wedge{}_{\mu\nu}^{(p)}} = Q^{\wedge\mu\nu}$$

In that case, instead of (5.7), we obtain from (5.6)

$$Q^{\wedge\mu\nu} = p^{\wedge\mu\nu} - \rho \frac{\partial U}{\partial \varepsilon^\wedge{}_{\mu\nu}^{(p)}} \tag{5.10}$$

For a medium "with memory", (5.9) is replaced by(*)

$$\sigma_2 = \left(p^{\wedge\mu\nu} - \rho\,\frac{\partial U}{\partial \varepsilon^{\wedge(p)}_{\mu\nu}}\right) e^{\wedge(p)}_{\mu\nu} \tag{5.11}$$

The preceding discussion on σ_2 is essential, since (5.9) is widely used in the literature. The complete equation of entropy balance (4.1) becomes in the present case

$$\rho\Theta\,\frac{dS}{dt} = -\,\Theta\nabla^{\wedge}_{\mu}\,\frac{q^{\wedge\mu}}{\Theta} - \frac{q^{\wedge\mu}}{\Theta}\,\nabla^{\wedge}_{\mu}\Theta + Q^{\wedge\mu\nu}e^{\wedge(p)}_{\mu\nu} \tag{5.12}$$

The dissipation function σ is given by

$$\sigma = -\,\frac{q^{\wedge\mu}}{\Theta}\,\nabla^{\wedge}_{\mu}\Theta + \sigma_2, \qquad \sigma_2 = Q^{\wedge\mu\nu}e^{\wedge(p)}_{\mu\nu} \tag{5.13}$$

Formula (5.13) shows that only for isothermic states is the requirement $\sigma_2 \geqslant 0$ completely substantiated.

From the condition that the entropy increase must be independent of deformation time, we obtain the result that σ_2, which is a homogeneous first-degree function in $e^{\wedge(p)}_{\mu\nu}$, may also depend on the appropriate hardening parameters χ_s.

From the theory developed in Section 4, it follows that for irreversible processes

$$Q^{\wedge\mu\nu} = \frac{\partial\sigma_2}{\partial e^{\wedge(p)}_{\mu\nu}} \tag{5.14}$$

Relations (5.14) together with (5.5) may be regarded as equations defining the plastic deformation rate tensor $e^{\wedge(p)}_{\mu\nu}$.

These relations replace the associated rule. After determining the loading functions from (5.14), we may obtain [25] the associated rule which generally contains the components $Q^{\wedge\mu\nu}$ but not components of the stress tensor $p^{\wedge\mu\nu}$, as it was assumed in [25].

Note that in the case under consideration relations of the type given in (5.14) may be written for $p^{\wedge\mu\nu}$ as well as $Q^{\wedge\mu\nu}$:

$$p^{\wedge\mu\nu} = \frac{\partial\varphi}{\partial e^{\wedge(p)}_{\mu\nu}}$$

However, the function $\varphi \gtrless 0$ differs from the dissipation function σ_2, and it follows from (5.10), (5.14) and (5.4) that the relation between the two is given by

$$\varphi = \sigma_2 + \rho\,\frac{\partial U}{\partial \varepsilon^{\wedge(p)}_{\mu\nu}}\,e^{\wedge(p)}_{\mu\nu} + \omega$$

where ω is an arbitrary function which is independent of $e^{\wedge(p)}_{\mu\nu}$.

For a medium "with memory" in accordance with (5.10), $Q^{\wedge\mu\nu}$ may be replaced by $p^{\wedge\mu\nu}$ as arguments of the loading functions. For reversible processes, we may take $\sigma_2 = 0$. In many typical cases, it follows from $\sigma_2 = 0$ that $e^{\wedge(p)}_{\mu\nu} = 0$. However, we can consider functions σ_2 such that residual or plastic deformations may arise for reversible processes.

Given suitable forms of the internal energy and the dissipation function, we may obtain from (3.2), (5.12) and (5.14) concrete models in the theory of plasticity for which the hardening parameters χ_s are given functions or functionals of the defining parameters. If the dissipation function σ_2 is of the form(**)

$$\sigma_2 = k\sqrt{e_{\mu\nu}^{(p)}e^{(p)\mu\nu}}$$

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*) An explanation of the distinction and relation between the dissipation and the work due to the stress tensor for plastic deformations is given in [14] (p. 269).

**) For such a choice of σ_2, the plastic deformation rate tensor may be replaced by the deviator of the plastic deformation rate tensor. The corresponding changes in the resultant formulas are evident.

where k is independent of $e_{\mu\nu}^{(p)}$, then it is easily seen that the $Q^{\wedge\mu\nu}$, defined by (5.14), lie on the surface

$$Q^{\wedge\mu\nu}Q^{\wedge}{}_{\mu\nu} = k^2 \tag{5.15}$$

In the ideal plastic model of von Mises [21]

$$U = U(\varepsilon^{\wedge(e)}_{\mu\nu}, S, \mathbf{L}_{(p)}), \qquad k = \text{const}$$

In the Schmidt-Osgood plastic models with hardening, we have

$$U = U(\varepsilon^{\wedge(e)}_{\mu\nu}, S, \mathbf{L}_{(p)}), \qquad k = k(\chi)$$

where χ is defined by the relation $d\chi = \sqrt{d\varepsilon_{\mu\nu}^{(p)} d\varepsilon^{(p)\mu\nu}}$ and $k(\chi)$ is an empirically determined function. Since the internal energy is, in these cases, independent of the plastic deformations, (5.7) holds and the equation for the loading surface may be written in the form

$$p^{\wedge\mu\nu}p^{\wedge}{}_{\mu\nu} = k^2$$

For models with translation hardening, we set

$$U = U_0(\varepsilon^{\wedge(e)}_{\mu\nu}, S, \mathbf{L}_{(p)}) + {}^1/_2 c\varepsilon^{\wedge(p)}_{\mu\nu}\varepsilon^{\wedge(p)\mu\nu}, \qquad k = \text{const}$$

Here, instead of (5.7), we obtain

$$Q^{\wedge\mu\nu} = p^{\wedge\mu\nu} - c\varepsilon^{\wedge(p)\mu\nu}$$

Thus, the loading surface equation (5.15) takes the form

$$(p^{\wedge\mu\nu} - c\varepsilon^{\wedge(p)\mu\nu})(p^{\wedge}{}_{\mu\nu} - c\varepsilon^{\wedge(p)}_{\mu\nu}) = k^2$$

6. Example model in the theory of continuous dislocations. The variational principle permits the construction of various models of media with continuously distributed dislocations. Below, we examine a particular example within the confines of small deformation theory(*); in this model, the kinetic energy is related only to the inertial properties of the actual state $T = {}^1\!/_2\rho v^2$, while the internal energy is a quadratic form in the components of the elastic strain tensor $\varepsilon_{\mu\nu}^{(e)}$, in the entropy difference $S - S_0$ between the deformed and initial states, and, by contrast with classical elasticity theory, in nine other internal parameters which are components of the dislocation density tensor $S^{\alpha\beta}$:

$$\begin{aligned} U = {}^1/_2 A^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}^{(e)}\varepsilon_{\gamma\delta}^{(e)} + B^{\alpha\beta}\varepsilon_{\alpha\beta}^{(e)}(S - S_0) + C^{\alpha\beta}{}_{\gamma\delta}\varepsilon_{\alpha\beta}^{(e)}S^{\gamma\delta} + \\ + D_{\alpha\beta}S^{\alpha\beta}(S - S_0) + {}^1/_2 E_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta} + f(S) \end{aligned} \tag{6.1}$$

where $A^{\alpha\beta\gamma\delta}, \ldots, E_{\alpha\beta\gamma\delta}$ are given nonvarying parameters (in the general theory, they correspond to the parameters $\mathbf{L}_{(p)}$). An additive constant may be taken as a function of the entropy $f(S)$.

Set

$$\Lambda = T - \rho U$$

$$\delta W^* = \int_V \int_t [\rho\Theta\delta S + F_\alpha \delta_1 x^\alpha - Q^{\mu\nu}\delta\varepsilon_{\mu\nu}^{(p)} - Q^{\mu\nu\lambda}\nabla_\lambda (B_{\mu\alpha}\delta A^\alpha{}_\nu) - \operatorname{div} \mathbf{q}\delta t]\, d\tau\, dt \tag{6.2}$$

In correspondence with the general theory of Section 5, the components $Q^{\mu\nu}$ and q^α are defined by (4.3). The dissipation functions are given by the following Expressions:

$$\sigma_1 = \Theta^{-1}F^{\alpha\beta}(\nabla_\alpha\Theta)(\nabla_\beta\Theta) \tag{6.3}$$

$$\sigma_2 = \left(K^{\alpha\beta\gamma\delta}e_{\alpha\beta}^{(p)}e_{\gamma\delta}^{(p)} + 2L^{\alpha\beta\gamma\delta}e_{\alpha\beta}^{(p)}\frac{dS_{\gamma\delta}}{dt} + M^{\alpha\beta\gamma\delta}\frac{dS_{\alpha\beta}}{dt}\frac{dS_{\gamma\delta}}{dt}\right)^{1/2} \tag{6.4}$$

*) As we know, in the theory of small deformations the components of the strain tensor in the observer reference system and those in the moving reference system are identical. As a result, we will omit in the following the symbol $^\wedge$, and we will not distinguish between operators D and $d(\ldots)/dt$, $\partial(\ldots)/\partial\xi^\mu$ and $\partial(\ldots)/\partial x^\alpha$ when they are applied to components of small tensors. Moreover it is assumed that $g^\circ_{\mu\nu}(\xi^\lambda) = g_{\mu\nu}(\xi^\lambda)$.

Here, $F^{\alpha\beta}$, $K^{\alpha\beta\gamma\delta}$, $L^{\alpha\beta\gamma\delta}$ and $M^{\alpha\beta\gamma\delta}$ are components of physical properties of the medium which, in view of the positive definite character of σ_1 and σ_2, must satisfy the known inequalities. Utilizing (4.3), we find that, for small deformations, the following Eqs. hold:

$$dS_{\alpha\beta} / dt = {}^1/_2\, \varepsilon^{\gamma\sigma}{}_{\alpha} \nabla_\gamma \pi_{\beta\sigma} \tag{6.5}$$

From (4.3) and (6.3) to (6.5), it follows that

$$q^\alpha = - F^{\alpha\beta} \nabla_\beta \Theta \tag{6.6}$$

$$Q^{\alpha\beta} = Q^{\beta\alpha} = \frac{\partial \sigma_2}{\partial e_{\alpha\beta}^{(p)}} = \frac{1}{\sigma_2} \left(K^{\alpha\beta\gamma\delta} e_{\gamma\delta}^{(p)} + L^{\alpha\beta\gamma\delta} \frac{dS_{\gamma\delta}}{dt} \right) \tag{6.7}$$

$$Q^{\beta\sigma\gamma} = \frac{\partial \sigma_2}{\partial (\nabla_\gamma \pi_{\beta\sigma})} = {}^1/_2 \frac{\partial \sigma_2}{\partial (dS_{\alpha\beta} / dt)} \varepsilon^{\gamma\sigma}{}_{\alpha} = {}^1/_2 R^{\alpha\beta} \varepsilon^{\gamma\sigma}{}_{\alpha} \tag{6.8}$$

From (6.8), we obtain eighteen relations

$$Q^{\beta\sigma\gamma} + Q^{\beta\gamma\sigma} = 0 \tag{6.9}$$

which are part of (4.7), defining a loading surface in the space $\{Q^{\alpha\beta}, Q^{\alpha\beta\gamma}\}$. The remaining nine components $Q^{\beta[\sigma\gamma]}$ may be expressed in terms of the nine components of the tensor

$$R^{\alpha\beta} = \frac{\partial \sigma_2}{\partial (dS_{\alpha\beta} / dt)} = \frac{1}{\sigma_2} \left(L^{\gamma\delta\alpha\beta} e_{\alpha\beta}^{(p)} + M^{\alpha\beta\gamma\delta} \frac{dS_{\gamma\delta}}{dt} \right) \tag{6.10}$$

In the space $\{Q^{\alpha\beta}, R^{\alpha\beta}\}$, the equation of the loading surface takes the form

$$f(Q^{\alpha\beta}, R^{\alpha\beta}, K^{\alpha\beta\gamma\delta}, L^{\alpha\beta\gamma\delta}, M^{\alpha\beta\gamma\delta}) = 0 \tag{6.11}$$

In the case under consideration, we obtain from the general theory, Sections 4 and 5, the following system of equations, consisting of the equations of motion (3.9), entropy balance

$$\rho\Theta \frac{dS}{dt} = - \Theta \nabla_\alpha \frac{q^\alpha}{\Theta} + \sigma_1 + \sigma_2$$

Eqs. for the internal parameters (3.4)

$$ {}^1/_2 \varepsilon^{\gamma\beta}{}_{\sigma} \nabla_\gamma (R^{\sigma\alpha} + \Sigma^{\sigma\alpha}) + p^{\alpha\beta} = Q^{\alpha\beta} \tag{6.12}$$

Eqs. of state

$$\rho^{-1} p^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}^{(e)} + B^{\alpha\beta} (S - S_0) + C^{\alpha\beta}{}_{\gamma\delta} S^{\gamma\delta} \tag{6.13}$$

$$\rho^{-1} \Sigma_{\gamma\delta} = C^{\alpha\beta}{}_{\gamma\delta} \varepsilon_{\alpha\beta}^{(e)} + D_{\gamma\delta} (S - S_0) + E_{\alpha\beta\gamma\delta} S^{\gamma\delta} \tag{6.14}$$

as well as (6.6), (6.7) and (6.10). Eqs. (6.12), symmetrized with respect to $\alpha\beta$, differ from (5.7) of the theory of plasticity by the term $1/2\, \varepsilon^{\gamma\beta}{}_{\sigma} \nabla_\gamma (R^{\sigma\alpha} + \Sigma^{\sigma\alpha})$. Eqs. (6.12), alternated with respect to $\alpha\beta$ take the form

$$\nabla_\beta (R'^{\,\beta}_{\alpha} + \Sigma'^{\,\beta}_{\alpha}) - {}^2/_3 \nabla_\alpha (R_\beta{}^\beta + \Sigma_\beta{}^\beta) = 0 \tag{6.15}$$

Here, the primed quantities denote the respective deviators. In problems for which homogeneity of the particles exists, i.e. when $\nabla^{\wedge}_{\nu} A^{\alpha}_{\mu} = 0$ and $\nabla^{\wedge}_{\nu} S = 0$, we obtain $S^{\mu\nu} = 0$ so that the solution for the proposed model coincides with the solution obtained for the same problem by considering the model to be that of a plastic medium "with no memory" as discussed in Section 6.

Relations (6.12) may be utilized to replace $Q^{\alpha\beta}$ by $p^{\alpha\beta}$, $R^{\sigma\alpha}$ and $\Sigma^{\sigma\alpha}$ by $\nabla_\gamma R^{\sigma\alpha}$ and $\nabla_\gamma \Sigma^{\sigma\alpha}$ in (6.11). Hence it is clear that it is convenient to operate with $Q^{\alpha\beta}$ in place of $p^{\alpha\beta}$ when formulating the loading conditions.

(Note that taking into account the kinetic energy and inertial properties of the initial state would lead to a complication of (6.15) by terms of the form dK_α/dt where K_α is a generalized impulse associated with the motion of dislocations).

Thus, we have a general linear theory, taking into account anisotropy. All preceding formulas become substantially simplified if anisotropic properties manifest themselves [22] only through $e^{(p)}_{\alpha\beta}$ and $S^{\alpha\beta}$. In that case, instead of (6.6), (6.7) and (6.10) we obtain

$$q^{\alpha} = -\varkappa \nabla^{\alpha}\Theta \tag{6.16}$$

$$Q_{\alpha\beta} = \frac{1}{\sigma_2}\left[l_1 e^{(p)}_{\alpha\beta} + \left(l_2 e^{(p)}_{\gamma\delta} + l_3 \frac{dS_{\gamma\delta}}{dt}\right) g^{\gamma\delta} g_{\alpha\beta} + l_4 \frac{dS_{(\alpha\beta)}}{dt}\right]$$

$$R_{\alpha\beta} = \frac{1}{\sigma_2}\left[l_4 e_{\alpha}{}^{(p)}_{\beta} + \left(l_3 e^{(p)}_{\beta}{}_{\gamma} + l_5 \frac{dS_{\gamma\delta}}{dt}\right) g^{\gamma\delta} g_{\alpha\beta} + l_6 \frac{dS_{\alpha\beta}}{dt} + l_7 \frac{dS_{\beta\alpha}}{dt}\right]$$

where $\varkappa$, $l_1, \ldots, l_7$ are scalars which in the general case may be considered as functions of the scalar invariants of the defining parameters, while in the linear theory they may be taken as constants. The loading function (6.11), corresponding to the dissipation function (6.4) may be obtained by substituting into the right-hand side of the following Eq.

$$\sigma_2 = Q^{\alpha\beta} e^{(p)}_{\alpha\beta} + Q^{\alpha\beta\gamma} \nabla_{\gamma} \pi_{\alpha\beta} = Q^{\alpha\beta} e_{\alpha\beta}^{(p)} + R^{\alpha\beta} \frac{dS_{\alpha\beta}}{dt}$$

the quantities $e^{(p)}_{\alpha\beta}$ and $dS_{\alpha\beta}/dt$, which from (6.16) may be experssed in terms of $\sigma_2 Q^{\alpha\beta}$ and $\sigma_2 R^{\alpha\beta}$ and which appear in the form

$$f(Q^{\alpha\beta}, R^{\alpha\beta}, l_1, \ldots, l_7) = (Q'_{\alpha\beta} - c_1 R_{\alpha\beta})(Q'^{\alpha\beta} - c_1 R^{\alpha\beta}) - c_2 R'_{(\alpha\beta)} R'^{(\alpha\beta)} -$$
$$- c_3 R_{[\alpha\beta]} R^{[\alpha\beta]} - c_4 (g_{\alpha\beta} R^{\alpha\beta})^2 - c_5 (g_{\alpha\beta} Q^{\alpha\beta})^2 - c_6 (g_{\alpha\beta} R^{\alpha\beta})(g_{\alpha\beta} Q^{\alpha\beta}) - c_0 \tag{6.17}$$

where c_0, $c_1, \ldots c_6$ are defined in terms of $l_1, \ldots, l_7$ in an obvious manner. The loading function (6.17) is a second order surface in the space $\{Q^{\alpha\beta}, R^{\alpha\beta}\}$. If c_i = const., this surface is fixed. If $c_5 = c_6 = 0$, then the plastic deformations are incompressible. The case of l_1 = const., $l_2 = l_3 = \ldots = l_7 = 0$ ($c_1 = \ldots = c_6 = 0$, c_0 = const.) corresponds to an ideally plastic body, studied in Section 5, in which case $R^{\alpha\beta} = 0$.

In the general case, $R^{\alpha\beta}$ is nonzero and describes the hardening phenomenon. In the stress space $\{Q'^{\alpha\beta}\}$, the loading surface is a sphere whose center $c_1 R^{\alpha\beta}$ varies with time and whose radius r

$$r^2 = c_0 + c_2 R'_{(\alpha\beta)} R'^{(\alpha\beta)} + c_3 R_{[\alpha\beta]} R^{[\alpha\beta]} + c_4 (g_{\alpha\beta} R^{\alpha\beta})^2$$

also varies during the process of deformation. Concrete models are obtained by a choice of the coefficients l_i (or c_i).

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